



Liebeck, M. W., & Thomas, A. (2017). Finite subgroups of simple algebraic groups with irreducible centralizers. *Journal of Group Theory*, 20(5), 841-870. <https://doi.org/10.1515/jgth-2017-0002>

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# Finite subgroups of simple algebraic groups with irreducible centralizers

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January 6, 2017

## Abstract

We determine all finite subgroups of simple algebraic groups that have irreducible centralizers – that is, centralizers whose connected component does not lie in a parabolic subgroup.

## 1 Introduction

Let  $G$  be a simple algebraic group over an algebraically closed field. Following Serre [15], a subgroup of  $G$  is said to be  $G$ -irreducible (or just *irreducible* if the context is clear) if it is not contained in a proper parabolic subgroup of  $G$ . Such subgroups necessarily have finite centralizer in  $G$  (see [13, 2.1]). In this paper we address the question of which finite subgroups can arise as such a centralizer. It turns out (see Corollary 4 below) that they form a very restricted collection of soluble groups, together with the alternating and symmetric groups  $Alt_5$  and  $Sym_5$ .

The question is rather straightforward in the case where  $G$  is a classical group (see Proposition 3 below). Our main result covers the case where  $G$  is of exceptional type.

**Theorem 1** *Let  $G$  be a simple adjoint algebraic group of exceptional type in characteristic  $p \geq 0$ , and suppose  $F$  is a finite subgroup of  $G$  such that  $C_G(F)^0$  is  $G$ -irreducible. Then  $|F|$  is not divisible by  $p$ , and  $F, C_G(F)^0$  are as in Tables 7–12 in Section 5 at the end of the paper (one  $G$ -class of subgroups for each line of the tables).*

**Remarks** (1) The notation for the subgroups  $F$  and  $C_G(F)^0$  is described at the end of this section; the notation for elements of  $F$  is defined in Proposition 2.2.

- (2) The theorem covers adjoint types of simple algebraic groups. For other types, the possible finite subgroups  $F$  are just preimages of those in the conclusion.
- (3) We also cover the case where  $G = \text{Aut } E_6 = E_6.2$  (see Table 10 and Section 3.3).
- (4) A complete determination of all  $G$ -irreducible connected subgroups is carried out in [16].

Every finite subgroup  $F$  in Theorem 1 is contained in a maximal such finite subgroup. The list of maximal finite subgroups with irreducible centralizers is recorded in the next result.

**Corollary 2** *Let  $G$  be a simple adjoint algebraic group of exceptional type, and suppose  $F$  is a finite subgroup of  $G$  which is maximal subject to the condition that  $C_G(F)^0$  is  $G$ -irreducible. Then  $F$ ,  $C_G(F)^0$  are as in Table 1.*

For the classical groups we prove the following.

**Proposition 3** *Let  $G$  be a classical simple algebraic group in characteristic  $p \geq 0$  with natural module  $V$ , and suppose  $F$  is a finite subgroup of  $G$  such that  $C_G(F)^0$  is  $G$ -irreducible. Then  $p \neq 2$  and  $F$  is an elementary abelian 2-group. Moreover,  $G \neq SL_n$  and the following hold.*

(i) *If  $G = Sp_{2n}$ , then*

$$C_G(F)^0 = \prod_i Sp_{2n_i} = \prod_i Sp(W_i),$$

*where  $\sum n_i = n$  and  $W_i$  are the distinct weight spaces of  $F$  on  $V$ .*

(ii) *If  $G = SO_n$ , then*

$$C_G(F)^0 = \prod_i SO_{n_i} = \prod_i SO(W_i),$$

*where  $n_i \geq 3$  for all  $i$ ,  $\sum n_i = n$  or  $n - 1$ , and  $W_i$  are weight spaces of  $F$ .*

**Remark** In Section 4 we prove a version of this result covering finite subgroups of  $\text{Aut } G$  for  $G$  of classical type.

**Corollary 4** *Let  $G$  be a simple adjoint algebraic group, and let  $D$  be a proper connected  $G$ -irreducible subgroup. Then the finite group  $C_G(D)$  is either elementary abelian or isomorphic to a subgroup of one of the following groups:*

$$2_-^{1+4}, G_{12}, \text{Sym}_4 \times 2, SL_2(3), 3^2.\text{Dih}_8, \text{Sym}_5.$$

**Notation** Throughout the paper we use the following notation for various finite groups:

$Z_n$ , or just $n$	cyclic group of order $n$
$p^s$ ( $p$ prime)	elementary abelian group of order $p^s$
$Alt_n, \text{Sym}_n$	alternating and symmetric groups
$\text{Dih}_{2n}$	dihedral group of order $2n$
$4 \circ \text{Dih}_8$	order 16 central product with centre $Z_4$
$2_-^{1+4}$	extra-special group of order 32 of minus type
$\text{Frob}_{20}$	Frobenius group of order 20
$G_{12}$	dicyclic group $\langle x, y \mid x^6 = 1, x^y = x^{-1}, y^2 = x^3 \rangle$ of order 12

In the tables in Section 5, and also in the text, we shall sometimes use  $\bar{A}_1$  to denote a subgroup  $A_1$  of a simple algebraic group  $G$  that is generated by long root subgroups; we use the notation  $\bar{A}_2$  similarly. Also,  $B_r$  denotes a natural subgroup of type  $SO_{2r+1}$  in a group of type  $D_n$ .

We use the following notation when describing modules for a semisimple algebraic group  $G$ . We let  $L(G)$  denote the Lie algebra of  $G$ . If  $\lambda$  is a dominant weight, then

Table 1: Maximal finite subgroups  $F$  with irreducible centralizer

$G$	$F$	$p$	$C_G(F)^0$
$E_8$	$2^4$	$p \neq 2$	$A_1^8$
	$Q_8$	$p \neq 2$	$B_2^3$
	$2_-^{1+4}$	$p \neq 2$	$B_1^5$
	$Dih_6$	$p \neq 2, 3$	$B_4$
	$G_{12}$	$p \neq 2, 3$	$\bar{A}_1 A_1 A_3$
	$Sym_4 \times 2$	$p \neq 2, 3$	$\bar{A}_1 A_1 A_1$
	$SL_2(3)$	$p \neq 2, 3$	$\bar{A}_1 A_2$
	$3^2.Dih_8$	$p \neq 2, 3$	$A_1^2$
	$Sym_5$	$p \neq 2, 3, 5$	$A_1$
	$Q_8$	$p = 3$	$\bar{A}_1 D_4$
	$Dih_8 \times 2$	$p = 3$	$\bar{A}_1^2 B_1^2 B_2$
	$3$	$p = 2$	$A_8$
	$3^2$	$p = 2$	$A_2^4$
	$5$	$p = 2$	$A_4^2$
	$Frob_{20}$	$p = 3$	$B_2$
$E_7$	$2^3$	$p \neq 2$	$A_1^7$
	$Q_8$	$p \neq 2$	$\bar{A}_1 B_1^4$
	$Dih_6$	$p \neq 2, 3$	$A_1 A_3$
	$Alt_4$	$p \neq 2, 3$	$A_2$
	$Sym_4$	$p \neq 2, 3$	$\bar{A}_1 A_1$
	$2^2$	$p = 3$	$D_4$
	$Dih_8$	$p = 3$	$\bar{A}_1 B_1^2 B_2$
	$3$	$p = 2$	$A_2 A_5$
$E_6$	$2$	$p \neq 2$	$A_1 A_5$
	$Dih_6$	$p \neq 2, 3$	$A_1 A_1$
	$3$	$p = 2$	$A_2^3$
$F_4$	$2^3$	$p \neq 2$	$A_1^4$
	$Q_8$	$p \neq 2$	$B_1^3$
	$Sym_4$	$p \neq 2, 3$	$A_1$
	$3$	$p = 2$	$A_2 A_2$
	$Dih_8$	$p = 3$	$B_1 B_2$
$G_2$	$Dih_6$	$p \neq 2, 3$	$A_1$
	$2$	$p = 3$	$A_1 A_1$
	$3$	$p = 2$	$A_2$

Table 2: Centralizers of graph automorphisms in simple algebraic groups

$G$	order of $t$	$C_G(t)^0$
$A_{2n}$	2	$B_n (p \neq 2)$
$A_{2n-1}$	2	$C_n$
$D_n$	2	$D_n (p \neq 2)$
$D_4$	3	$B_{n-1}$
$E_6$	2	$B_k B_{n-k-1} (1 \leq k \leq n-2, p \neq 2)$
		$G_2$
		$A_2 (p \neq 3)$
		$F_4$
		$C_4 (p \neq 2)$

$V_G(\lambda)$  (or simply  $\lambda$ ) denotes the rational irreducible  $G$ -module with high weight  $\lambda$ . When  $G$  is simple the fundamental dominant weights  $\lambda_i$  are ordered with respect to the labelling of the Dynkin diagrams as in [3, p. 250]. If  $V_1, \dots, V_k$  are  $G$ -modules, then  $V_1 / \dots / V_k$  denotes a module having the same composition factors as  $V_1 + \dots + V_k$ . Finally, when  $H$  is a subgroup of  $G$  and  $V$  is a  $G$ -module we use  $V \downarrow H$  for the restriction of  $V$  to  $H$ .

## 2 Preliminaries

In this section we collect some preliminary results required in the proof of Theorem 1.

**Proposition 2.1** *Let  $G$  be a simple adjoint algebraic group in characteristic  $p$ . Suppose  $t \in \text{Aut } G \setminus G$  is such that  $t$  has prime order and  $C_G(t)^0$  is  $G$ -irreducible. Then  $t, C_G(t)^0$  are given in Table 2.*

*If  $G = D_4$  and  $t$  has order 3 with  $C_G(t) = A_2$ , there is an involutory graph automorphism of  $G$  that inverts  $t$  and acts as a graph automorphism on  $C_G(t)$ .*

*Proof.* The first part follows from [8, Tables 4.3.3, 4.7.1] for  $p \neq 2$ , from [1, §8] for  $G = D_n, p = 2$ , and from [1, 19.9] for  $G = A_n, E_6, p = 2$ . The last part follows from [9, 2.3.3]. ■

**Proposition 2.2** *Let  $G$  be a simple adjoint algebraic group of exceptional type in characteristic  $p$ , and let  $x \in G$  be a nonidentity element such that  $C_G(x)^0$  is  $G$ -irreducible. Then  $x$  and  $C_G(x)$  are as in Table 3; we label  $x$  according to its order, which is not divisible by  $p$ .*

*Proof.* First observe that if  $p$  divides the order of  $x$  then  $C_G(x)^0$  is  $G$ -reducible by [2, 2.5]. Hence  $x$  is a semisimple element and  $C_G(x)^0$  is a semisimple subgroup of maximal rank. It follows from [14, 4.5] that this implies the order of  $x$  is equal to one of the coefficients in the expression for the highest root in the root system of  $G$ ; these are at most 6 for  $G = E_8$ , and at most 4 for the other types. The classes and centralizers of elements of these orders can be found in [5, 3.1, 4.1] and [6, 3.1]. ■

We shall need a similar result for the group  $\text{Aut } E_6 = E_{6.2}$ .

Table 3: Elements of exceptional groups with irreducible centralizers

$G$	$x$	$C_G(x)$
$E_8$	$2A$	$A_1 E_7$
	$2B$	$D_8$
	$3A$	$A_8$
	$3B$	$A_2 E_6$
	$4A$	$A_1 A_7$
	$4B$	$A_3 D_5$
	$5A$	$A_4^2$
	$6A$	$A_1 A_2 A_5$
$E_7$	$2A$	$A_1 D_6$
	$2B$	$A_7.2$
	$3A$	$A_2 A_5$
	$4A$	$A_1 A_3^2.2$
$E_6$	$2A$	$A_1 A_5$
	$3A$	$A_2^3.3$
$F_4$	$2A$	$B_4$
	$2B$	$A_1 C_3$
	$3A$	$A_2 A_2$
	$4A$	$A_1 A_3$
$G_2$	$2A$	$A_1 A_1$
	$3A$	$A_2$

Table 4: Elements with irreducible centralizers in  $G = \text{Aut } E_6$

$x$	$C_{G'}(x)$
$2B$	$F_4$
$2C$	$C_4 (p \neq 2)$
$4A$	$A_1 A_3 (p \neq 2)$
$6A$	$A_2 A_2 (p \neq 3)$

**Proposition 2.3** *Let  $G = \text{Aut } E_6$  and let  $x \in G \setminus G'$  be such that  $C_G(x)^0$  is  $G'$ -irreducible. Then  $x$  and  $C_{G'}(x)$  are as in Table 4.*

*Proof.* If  $x$  is an involution then the result follows from Proposition 2.1. Suppose now that  $1 \neq x^2 \in G'$ . Then  $x^2$  has order 2 or 3 and  $C_{G'}(x^2) = A_1 A_5$  or  $A_2^3.3$ , respectively, by Proposition 2.2. In the former case  $x$  acts as a graph automorphism on the  $A_5$  factor and  $C_{G'}(x) = A_1 C_3$  or  $A_1 A_3$  by [8, Table 4.3.1]. Here  $A_1 C_3$  is not possible since this lies in a subgroup  $F_4$  and hence centralizes an involution in  $G \setminus G'$ .

In the case where  $C_{G'}(x^2) = A_2^3.3$ , the element  $x^2$  is of order 3 in  $C_{G'}(x^3)$ . By Proposition 2.2, the latter group must be  $F_4$  and  $C_{F_4}(x^2) = A_2 A_2$ . ■

We also require information on normalizers of certain maximal rank subgroups. The following proposition can be deduced from [4, Tables 7–11] and direct calculation in the Weyl groups of exceptional algebraic groups; many of the results can be found

Table 5: Normalizers of maximal rank subgroups of  $G$

$G$	$M$	$N_G(M)/M$
$E_8$	$A_8$	2
	$A_2E_6$	2
	$A_1A_7$	2
	$A_4^2$	4
	$D_4^2$	$Sym_3 \times 2$
	$A_1^4D_4$	$Sym_4$
	$A_2^4$	$GL_2(3)$
	$A_1^8$	$AGL_3(2)$
$E_7$	$A_2A_5$	2
	$A_1^7$	$GL_3(2)$
$E_6$	$A_2^3$	$Sym_3 \times 2$
$F_4$	$A_2A_2$	2
	$D_4$	$Sym_3$
$G_2$	$A_2$	2

in [12, Chapter 11].

**Proposition 2.4** *Let  $G$  be a simple algebraic group of exceptional type. Then Table 5 gives the groups  $N_G(M)/M$  (or  $N_{\text{Aut } G}(M)/M$  for  $G$  of type  $E_6$ ) for the given maximal rank subgroups  $M$  of  $G$ .*

Next we have a result about the Spin group  $Spin_n$  in characteristic  $p \neq 2$ . Recall that the centre of  $Spin_n$  is  $2^2$  if  $n$  is divisible by 4 and is  $Z_2$  if  $n$  is odd. In the former case the quotients of  $Spin_n$  by the three central subgroups of order two are  $SO_n$  and the two half-spin groups  $HSpin_n$ .

**Proposition 2.5** *Let  $G$  be  $HSpin_n$  (where  $4|n$ ) or  $Spin_n$  ( $n$  odd), in characteristic  $p \neq 2$ . Let  $\langle t \rangle = Z(G)$  (so that  $G/\langle t \rangle = PSO_n$ ) and suppose  $F$  is a finite 2-subgroup of  $G$  containing  $t$  such that  $C_G(F)^0$  is  $G$ -irreducible. Then the preimage of  $F/\langle t \rangle$  in  $SO_n$  is elementary abelian. Moreover, an element  $e \in F$  has order 2 if and only if its preimage in  $SO_n$  has  $-1$ -eigenspace of dimension divisible by 4.*

*Proof.* If the preimage contains an element  $e$  of order greater than 2, then  $C_{SO_n}(e)$  has a nontrivial normal torus, and hence  $C_G(F)^0$  cannot be irreducible. The assertion in the last sentence is well known. ■

In the following statement, by a *pure* subgroup of  $G$  we mean a subgroup all of whose nonidentity elements are  $G$ -conjugate.

**Proposition 2.6** *Let  $G = E_8$  in characteristic  $p$ .*

- (i) *If  $p \neq 2$  then  $G$  has two conjugacy classes of subgroups  $E \cong 2^2$  such that  $C_G(E)^0$  is  $G$ -irreducible, and one class of pure subgroups  $E \cong 2^3$ ; these are as*

follows:

$E$	elements	$C_G(E)^0$
$2^2$	$2B^3$	$D_4^2$
	$2A^2, 2B$	$A_1^2 D_6$
$2^3$	$2B^7$	$A_1^8$

Further,  $G$  has no pure subgroup  $2^4$ .

- (ii) If  $p \neq 3$  then  $G$  has one class of subgroups  $E \cong 3^2$  such that  $C_G(E)^0$  is  $G$ -irreducible. For this class,  $C_G(E) = A_2^4$ .

*Proof.* Part (i) follows from [5, 3.7, 3.8]. For (ii), let  $E = \langle x, y \rangle < G$  with  $E \cong 3^2$  and  $C_G(E)^0$  irreducible. Then  $C_G(x) \neq A_8$ , so Proposition 2.2 implies that  $C_G(x) = A_2 E_6$ , and also that  $C_{A_2 E_6}(y)^0 = A_2^4$ , as required. ■

The next result is taken from [13, Lemma 2.2].

**Proposition 2.7** *Suppose  $G$  is a classical simple algebraic group in characteristic  $p \neq 2$ , with natural module  $V$ . Let  $X$  be a semisimple connected subgroup of  $G$ . If  $X$  is  $G$ -irreducible then either*

- (i)  $G = A_n$  and  $X$  is irreducible on  $V$ , or
- (ii)  $G = B_n, C_n$  or  $D_n$  and  $V \downarrow X = V_1 \perp \dots \perp V_k$  with the  $V_i$  all nondegenerate, irreducible and inequivalent as  $X$ -modules.

### 3 Proof of Theorem 1

#### 3.1 The case $G = E_8$

We now embark on the proof of Theorem 1 for the case  $G = E_8$ . Let  $F$  be a finite subgroup of  $G$  such that  $C_G(F)^0$  is  $G$ -irreducible. Then  $C_G(F)^0$  is semisimple (see [13, 2.1]). Moreover  $C_G(E)^0$  is irreducible for all nontrivial subgroups  $E$  of  $F$ . Also  $F$  is a  $\{2, 3, 5\}$ -group by Proposition 2.2.

**Lemma 3.1** *If  $F$  is an elementary abelian 2-group, then  $F$  is as in Table 7.*

*Proof.* Suppose  $F \cong 2^k$ . If  $k \leq 2$ , or if  $k = 3$  and  $F$  is pure, the conclusion follows from Proposition 2.6(i).

Now assume that  $k = 3$  and  $F$  is not pure. By considering the  $2^2$  subgroups of  $F$ , all of which must be as in (i) of Proposition 2.6, we see that one of these, say  $\langle e_1, e_2 \rangle$ , is  $2B$ -pure, so that  $C_G(e_1, e_2)^0 = D_4^2$ . We have  $C_G(e_1)/\langle e_1 \rangle \cong PSO_{16}$ , and consider the preimage of  $F/\langle e_1 \rangle$  in  $SO_{16}$ . This preimage is elementary abelian by Proposition 2.5, so can be diagonalized, and we can take  $e_2 = (-1^8, 1^8)$ . Let  $e_3$  be a further element of  $F$  that is in class  $2A$ . Then the  $-1$ -eigenspace of  $e_3$  has dimension 4 or 12, and so the fact that  $C_G(F)^0$  is  $G$ -irreducible means that we can take  $e_3 = (-1^4, 1^4, 1^8)$ , so that  $C_{SO_{16}}(F)^0 = SO_4 SO_4 SO_8$ , and so  $C_G(E)^0 = A_1^4 D_4$  as in Table 7.

Next suppose  $k \geq 4$ . Then  $F$  is not pure by Proposition 2.6(i), so  $F$  contains a subgroup  $\langle e_1, e_2, e_3 \rangle \cong 2^3$  as in the previous paragraph. Arguing as above, we



can take a further element  $e_4$  of  $F$  to be  $(1^8, -1^4, 1^4)$ , so that  $C_G(e_1, \dots, e_4)^0 = A_1^8$ . There is no possible further diagonal involution in  $F$  such that  $C_G(F)^0$  is irreducible, so  $k = 4$ .  $\blacksquare$

In view of the previous lemma, we assume from now on that  $F$  is not an elementary abelian 2-group. Hence if  $F$  is a 2-group, it has exponent 4 by Proposition 2.2.

**Lemma 3.2** *Suppose  $F$  is a 2-group, and has no element in the class  $4B$ . Then one of the following holds:*

- (i)  $F \cong Z_4$ , generated by an element in class  $4A$ , and  $C_G(F)^0 = A_1A_7$ ;
- (ii)  $F \cong Q_8$  with elements  $2A, 4A^6$ , and  $C_G(F)^0 = A_1D_4$ .

*In both cases  $F$  is as in Table 7.*

*Proof.* Let  $e \in F$  have order 4. By hypothesis,  $e$  is in class  $4A$ , so  $C_G(e) = A_1A_7$ . There is nothing to prove if  $F \cong Z_4$ , so assume  $|F| > 4$  and pick  $f \in N_F(\langle e \rangle) \setminus \langle e \rangle$ . If  $f \in A_1A_7$  then  $C_G(e, f)$  has a normal torus, so  $e^f = e^{-1}$  and  $f$  induces an involutory graph automorphism on  $A_7$  (see Proposition 2.4). Hence  $C_{A_7}(f)^0 = C_4$  or  $D_4$  by Proposition 2.1. The subgroup  $C_4$  lies in a Levi subgroup  $E_6$  of  $C_G(A_1) = E_7$  (see the proof of [7, 2.15]), so the irreducibility of  $C_G(F)^0$  implies that  $C_{A_7}(f)^0 = D_4$ , hence  $C_G(e, f)^0 = A_1D_4$ . Also  $\langle e, f \rangle \cong Q_8$ , as shown in the proof of [7, 2.15]. Finally, if  $N_F(\langle e, f \rangle) > \langle e, f \rangle$  then some element of order 4 in  $\langle e, f \rangle$  has centralizer in  $F$  of order greater than 4, which we have seen to be impossible above. Hence  $F = \langle e, f \rangle$ .  $\blacksquare$

**Lemma 3.3** *Suppose  $F$  is a 2-group and has an element  $e$  in the class  $4B$ . If  $C_F(e) \neq \langle e \rangle$ , then  $F$  is as in Table 7 (one of the entries  $4 \times 2$ ,  $Dih_8 \times 2$ ,  $4 \circ Dih_8$ ,  $Q_8 \times 2$ ,  $2_-^{1+4}$ ).*

*Proof.* Assume that  $C_F(e) \neq \langle e \rangle$ . Then  $C_F(e)$  contains a group of order 8 in which  $e$  is central, and hence there is an involution  $e_1 \in C_F(e) \setminus \langle e \rangle$ . Diagonalizing the preimage of  $C_F(e^2) / \langle e^2 \rangle$  in  $SO_{16}$  (as in the proof of Lemma 3.1), we can take

$$e = (-1^6, 1^{10}), \quad e_1 = (1^6, -1^4, 1^6),$$

so that  $\langle e, e_1 \rangle \cong Z_4 \times Z_2$  and  $C_G(e, e_1)^0 = A_1^2A_3^2$ , as in the  $4 \times 2$  entry in Table 7.

Write  $E_0 = \langle e, e_1 \rangle$ . If there exists  $f \in (F \cap A_1^2A_3^2) \setminus E_0$ , then  $C_G(e, e_1, f)$  has a nontrivial normal torus, which is a contradiction. Hence  $F \cap C_G(E_0)^0 = E_0$ .

Suppose  $C_F(e) \neq E_0$ . If there is no involution in  $C_F(e) \setminus E_0$ , then  $C_F(e)$  contains a subgroup  $Z_4 \times Z_4$ , which is impossible. So let  $e_2$  be an involution in  $C_F(e) \setminus E_0$ . As  $e_2$  does not centralize  $e_1$ , we can take

$$e_2 = (1^6, -1, 1^3, -1^3, 1^3).$$

Then  $C_G(e, e_1, e_2)^0 = A_3B_1^3$  (where each  $B_1$  corresponds to a natural subgroup  $SO_3$  in  $SO_{16}$ ), and  $\langle e, e_1, e_2 \rangle = \langle e \rangle \circ \langle e_1, e_2 \rangle \cong 4 \circ Dih_8$ , as in Table 7. If  $F \neq E_1 := \langle e, e_1, e_2 \rangle$ , then there is an element  $e_3 \in N_F(E_1) \setminus E_1$ , and adjusting by an element of  $E_1$  we can take

$$e_3 = (-1^3, 1^3, -1, 1^9).$$

Then  $C_G(E_1, e_3)^0 = B_1^5$  and  $\langle E_1, e_3 \rangle \cong 2_-^{1+4}$ , as in Table 7. Finally, there are no possible further elements of  $F$  such that  $C_G(F)^0$  is irreducible.

Now suppose  $C_F(e) = E_0$  and  $F \neq E_0$ . Pick  $f \in N_F(E_0) \setminus E_0$ . Then  $f$  centralizes  $e^2$ , so we can diagonalize in the usual way; adjusting by an element of  $E_0$  and using the fact that  $C_G(E_0, f)$  has no nontrivial normal torus, we can take  $f \in \{e_4, e_5, e_6, e_7\}$ , where

$$\begin{aligned} e_4 &= (-1^3, 1^3, 1^4, -1, 1^5), & e_5 &= (-1^3, 1^3, 1^4, -1^3, 1^3), \\ e_6 &= (-1^3, 1^3, -1, 1^3, 1^6), & e_7 &= (-1, 1^5, 1^4, -1^3, 1^3). \end{aligned}$$

Let  $E_2 = \langle E_0, f \rangle$ .

If  $f = e_4$  then  $C_G(E_2)^0 = B_1^2 \bar{A}_1^2 B_2$  and  $E_2 = \langle e, e_4 \rangle \times \langle e_1 \rangle \cong Dih_8 \times Z_2$ , as in Table 7. There are no possible further elements of  $F$  in this case.

If  $f = e_5$  then  $C_G(E_2)^0 = \bar{A}_1^2 B_1^4$  and  $E_2 = \langle e, e_5 \rangle \times \langle e_1 \rangle \cong Q_8 \times Z_2$ , as in Table 7. Any further element of  $F$  would centralize  $e^2$ , and hence would violate the fact that  $C_F(e) = E_0$ .

Finally, if  $f = e_6$  or  $e_7$ , then  $E_2$  is  $D_8$ -conjugate to  $\langle e, e_1, e_2 \rangle$  or  $\langle e, e_1, e_4 \rangle$ , cases considered previously. ■

**Lemma 3.4** *If  $F$  is a 2-group, then it is as in Table 7.*

*Proof.* Suppose  $F$  is a 2-group. In view of the previous two lemmas, we can assume that  $F$  contains an element  $e$  in the class  $4B$ , and that  $C_F(e) = \langle e \rangle$  – and indeed that  $F$  has no subgroup  $Z_4 \times Z_2$ . We can also assume that  $F \neq \langle e \rangle$ . Hence there exists  $f \in F$  such that  $e^f = e^{-1}$ . As usual we can diagonalize  $\langle e, f \rangle$ , and hence take  $e = (-1^6, 1^{10})$  and  $f \in \{f_1, f_2, f_3, f_4, f_5\}$ , where

$$\begin{aligned} f_1 &= (-1^3, 1^3, -1, 1^9), & f_2 &= (-1^3, 1^3, -1^3, 1^7), \\ f_3 &= (-1^3, 1^3, -1^5, 1^5), & f_4 &= (-1, 1^5, -1^3, 1^7), \\ f_5 &= (-1, 1^5, -1^5, 1^5). \end{aligned}$$

Moreover, the fact that  $F$  has no subgroup  $Z_4 \times Z_2$  implies that  $F = \langle e, f \rangle$ .

It is easily seen that the possibilities for  $F$  and  $C_G(F)^0$  are as follows:

$F$	$C_G(F)^0$
$\langle e, f_1 \rangle \cong Dih_8$	$B_1^2 B_4$
$\langle e, f_2 \rangle \cong Q_8$	$B_1^3 B_3$
$\langle e, f_3 \rangle \cong Dih_8$	$B_1^2 B_2^2$
$\langle e, f_4 \rangle \cong Dih_8$	$B_1 B_2 B_3$
$\langle e, f_5 \rangle \cong Q_8$	$B_2^3$

All these possibilities are in Table 7. ■

**Lemma 3.5** *If  $F$  is a 3-group, then  $F = 3$  or  $3^2$  is as in Table 7.*

*Proof.* Suppose  $F$  is a 3-group. It has exponent 3, by Proposition 2.2. If  $F = 3$  or  $3^2$ , it is as in Table 7 by Propositions 2.2 and 2.6(ii), so assume  $|F| > 9$ .

If  $F$  has an element  $e$  with  $C_G(e) = A_8$ , then there is an element  $f \in C_F(e) \setminus \langle e \rangle$ , and  $C_{A_8}(f)$  is reducible in  $A_8$ , a contradiction. Hence all nonidentity elements of  $F$

have centralizer  $A_2E_6$  (see Proposition 2.2). In particular, they have trace 5 on the adjoint module  $L(G)$  (see [5, 3.1]).

Let  $V$  be a normal subgroup of  $F$  with  $V \cong 3^2$ . Then  $C_G(V) = A_2^4$  by Proposition 2.6(ii). If  $f \in F \setminus V$ , then  $f$  acts as a 3-cycle on the four  $A_2$  factors, so  $C_G(V, f)^0 = \bar{A}_2A_2$ . On the other hand, since every nonidentity element has trace 5 on  $L(G)$ , we have

$$\dim C_{L(G)}(V, f) = \frac{1}{27}(248 + 26 \cdot 5) = 14.$$

This is a contradiction, showing that  $|F| > 9$  is impossible.  $\blacksquare$

From now on, we assume that  $F$  is not a 2-group or a 3-group. Let  $J = \text{Fit}(F)$ , the Fitting subgroup of  $F$ .

**Lemma 3.6** *Suppose  $J$  is a nontrivial 2-group. Then  $F$  is as in Table 7.*

*Proof.* By Lemma 3.4,  $J$  and  $C_G(J)^0$  are as in Table 7. Also  $C_F(J) \leq J$ , so  $F$  contains an element  $x$  of order  $r = 3$  or  $5$  acting nontrivially on  $J$  and as a graph automorphism of  $C_G(J)^0$ . By inspection of Table 7, the possibilities for  $J$  with these properties are as follows:

$J$	$C_G(J)^0$	$r$
$2^2$	$D_4^2$	3
$2^3$	$A_1^4D_4$	3
	$A_1^8$	3, 5
$2^4$	$A_1^8$	3, 5
$Q_8$	$A_1D_4$	3
	$B_2^3$	3
	$B_1^3B_3$	3
$4 \circ Dih_8$	$A_3B_1^3$	3
$Q_8 \times 2$	$A_1^2B_1^4$	3
$2_-^{1+4}$	$B_1^5$	3, 5

For the last five cases, where  $C_G(J)^0 \triangleright B_1^r$  (or  $B_2^r$ ),  $C_G(J, x)^0$  has a factor  $B_1$  (or  $B_2$ ) which is a diagonal subgroup of this, and so  $C_G(J, x)^0$  is a reducible subgroup of  $D_8$  in these cases, by Proposition 2.7. Also if  $C_G(J)^0 = A_1^8$ , then  $N_G(A_1^8)/A_1^8 \cong AGL_3(2)$  by Proposition 2.4, so  $r = 3$  and  $x$  acts as a product of two 3-cycles on the eight  $A_1$  factors. We claim that again  $C_G(J, x)^0$  is reducible. To see this, regard  $A_1^8$  as a subgroup of  $D_8$  corresponding to  $SO_4^4$  in  $SO_{16}$ . Observe that  $C_J(x)^0 = \bar{A}_1^2A_1^2$  where each of the last two  $A_1$  factors is diagonal in  $\bar{A}_1^3$ . There are two possible actions of the subgroup  $A_1^2 < \bar{A}_1^6 < D_6$  on the 12-dimensional natural module, namely  $(1, 1)^3$  or  $(2, 0) + (1, 1) + (0, 2) + (0, 0)^2$ . In both cases the subgroup  $A_1^2$  is  $D_6$ -reducible by Proposition 2.7 and hence  $C_J(x)^0$  is  $D_8$ -reducible.

This leaves the following possibilities remaining, all with  $r = 3$ :

$J$	$C_G(J)^0$
$2^2$	$D_4^2$
$2^3$	$A_1^4D_4$
$Q_8$	$A_1D_4$

Suppose  $J = 2^2$ ,  $C_G(J)^0 = D_4^2$ . Then  $x$  induces a triality automorphism on both  $D_4$  factors (see Proposition 2.4). So by Proposition 2.1,  $C_G(J, x)^0 = G_2G_2$ ,  $A_2A_2$

or  $A_2G_2$ . In the first case  $G_2G_2 < D_7 < D_8$ , so is reducible. Hence if  $F = \langle J, x \rangle$  we have the possibilities

$$F = \langle J, x \rangle \cong \text{Alt}_4, C_G(F)^0 = A_2A_2 \text{ or } A_2G_2,$$

both in Table 7. Now assume  $F \neq \langle J, x \rangle$ . Then  $F = \langle J, x, t \rangle \cong \text{Sym}_4$ , where  $t$  is an involution inverting  $x$  (since  $\text{Fit}(F) \cong 2^2$ ). If  $C_G(J, x)^0 = A_2A_2$  then  $t$  acts as a graph automorphism on each  $A_2$  factor (see Proposition 2.1), and so  $C_G(F)^0 = A_1A_1$ ; and if  $C_G(J, x)^0 = A_2G_2$  then  $t$  acts as a graph automorphism on the  $A_2$  factor and centralizes the  $G_2$  factor, so  $C_G(F)^0 = A_1G_2$ . Hence we have the possibilities

$$F = \langle J, x, t \rangle \cong \text{Sym}_4, C_G(F)^0 = A_1A_1 \text{ or } A_1G_2,$$

both in Table 7.

Next suppose  $J = 2^3$ ,  $C_G(J)^0 = A_1^4D_4$ . Then  $x$  acts as a 3-cycle on the  $A_1$  factors and as a triality on  $D_4$ , so  $C_G(J, x)^0 = \bar{A}_1A_1G_2$  or  $\bar{A}_1A_1A_2$  (where  $\bar{A}_1$  denotes a fundamental  $SL_2$  generated by a root group and its opposite). The first subgroup is reducible as it is contained in a subgroup  $D_7$  of  $D_8$ . So if  $F = \langle J, x \rangle$ , we have

$$F = \langle J, x \rangle \cong 2 \times \text{Alt}_4, C_G(F)^0 = \bar{A}_1A_1A_2,$$

as in Table 7. Now assume  $F \neq \langle J, x \rangle$ , and let  $\langle v \rangle = Z(\langle J, x \rangle)$ . As  $F/J$  is isomorphic to a subgroup of  $GL_3(2)$  with no nontrivial normal 2-subgroup, we have  $F/J \cong \text{Dih}_6$  and  $F = \langle J, x, t \rangle$  where  $x^t = x^{-1}$  and  $t^2 = 1$  or  $v$ . Such an element  $t$  centralizes both  $A_1$  factors of  $C_G(J, x)^0$ , and induces a graph automorphism on the  $A_2$  factor, so  $C_G(F)^0 = \bar{A}_1A_1A_1$ . This is contained in the above centralizer  $G_2A_2$  of an  $\text{Alt}_4$  subgroup, and  $\bar{A}_1A_1$  centralizes an involution in  $G_2$ . Hence in fact  $t^2 = 1$  and we have

$$F = \langle J, x, t \rangle \cong 2 \times \text{Sym}_4, C_G(F)^0 = \bar{A}_1A_1A_1,$$

as in Table 7.

Finally, suppose  $J = Q_8$ ,  $C_G(J)^0 = A_1D_4 < A_1A_7$ . Then  $x$  induces triality on the  $D_4$  factor (see [7, 2.15]), so  $C_G(J, x)^0 = A_1G_2$  or  $A_1A_2$ . The first subgroup is reducible in  $A_1A_7$ , so if  $F = \langle J, x \rangle$ , we have

$$F = \langle J, x \rangle = Q_8.3 \cong SL_2(3), C_G(F)^0 = \bar{A}_1A_2,$$

as in Table 7. If  $F \neq \langle J, x \rangle$  then  $F$  has an element  $t$  inducing a graph automorphism on the  $A_2$  factor, so  $C_G(J, x, t)^0 = A_1A_1$ , which is reducible in  $A_1A_7$ . This completes the proof.  $\blacksquare$

**Lemma 3.7** *If  $|J|_3 = 3$ , then  $F$  is as in Table 7.*

*Proof.* Assume  $|J|_3 = 3$ , and let  $x \in J$  be of order 3.

Suppose first that  $|F|_3 = 3$  also. As  $F$  has no element of order 15 we have  $|F|_5 = 1$ , so  $F/\langle x \rangle$  is a 2-group. The case where  $|F| = 3$  is in Table 7, so assume  $|F| > 3$ .

Suppose  $C_G(x) = A_2E_6$ . If  $t$  is an involution in  $C_F(x)$ , then  $C_G(x, t) = A_2A_1A_5$ ; moreover  $C_F(x)$  has no element of order 4 (as  $F$  has no element of order 12), and no subgroup  $V \cong 2^2$  (as  $C_G(x, V)$  would have a normal torus). Hence  $C_F(x)$  has order 3 or 6, and so  $|F|$  is 6 or 12.

If  $|F| = 6$ , then either  $F \cong Z_6$ ,  $C_G(F) = A_1A_2A_5$ , or  $F = \langle x, t \rangle \cong Dih_6$  with  $t$  inducing graph automorphisms on both factors of  $C_G(x) = A_2E_6$  (see Proposition 2.4), in which case  $C_G(F)^0 = A_1F_4$  or  $A_1C_4$ . All these possibilities are in Table 7.

If  $|F| = 12$ , then  $F = \langle y, u \rangle$  where  $y$  has order 6,  $y^u = y^{-1}$  and  $u^2 = 1$  or  $y^3$ . Then  $u$  induces a graph automorphism on the  $A_2, A_5$  factors of  $C_G(y) = A_1A_2A_5$ , so  $C_G(F)^0 = \bar{A}_1A_1C_3$  or  $\bar{A}_1A_1A_3$ . The subgroup  $\bar{A}_1A_1C_3$  is contained in  $A_1F_4$  and in  $A_1C_4$ , while the subgroup  $\bar{A}_1A_1A_3$  is contained in neither. Hence  $u$  is an involution in the first case, and has order 4 in the second. This gives the possibilities

$$F = \langle y, u \rangle \cong Dih_{12} \text{ or } G_{12}, \quad C_G(F)^0 = \bar{A}_1A_1C_3 \text{ or } \bar{A}_1A_1A_3 \text{ (resp.)},$$

both in Table 7.

Next suppose that  $C_G(x) = A_8$ . Then  $C_F(x) = \langle x \rangle$ , so  $F = \langle x, t \rangle \cong Dih_6$  where  $t$  induces a graph automorphism on  $A_8$ , giving  $C_G(F)^0 = B_4$ . This completes the case where  $|F|_3 = 3$ .

Finally, suppose  $|F|_3 = 3^2$ , and let  $\langle x, y \rangle$  be a Sylow 3-subgroup of  $F$ . Again,  $|F|_5 = 1$ . If  $F = \langle x, y \rangle$  then  $C_G(F)^0 = A_2^4$  by Proposition 2.6(ii), as in Table 7. Otherwise, as  $C_F(J) \leq J$  there must be a subgroup  $V \cong 2^2$  of  $J$  such that  $y$  acts nontrivially on  $V$ . But then  $C_G(x, V)^0 = C_{A_2E_6}(V)^0$  is reducible, a contradiction. ■

**Lemma 3.8** *If  $|J|_3 = 3^2$ , then  $F$  is as in Table 7.*

*Proof.* Let  $V \cong 3^2$  be a Sylow 3-subgroup of  $J$  (also of  $F$ , by Lemma 3.5). By Propositions 2.6(ii) and 2.4,  $C_G(V) = A_2^4$  and  $N_G(A_2^4)/A_2^4 \cong GL_2(3)$ . There is no involution in  $C_F(V)$ , so  $J = \text{Fit}(F) = V$  and  $F/J$  is a nontrivial 2-subgroup of  $GL_2(3)$ .

Suppose first that  $|F/J| = 2$ . There are two classes of involutions in  $GL_2(3)$ , with representatives  $i = -I$  and  $t = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ . Then  $i$  induces a graph automorphism on each  $A_2$  factor of  $C_G(V)$ , so  $C_G(V, i)^0 = A_1^4$ ; and  $t$  fixes two  $A_2$  factors, inducing a graph automorphism on one of them, so  $C_G(V, t)^0 = A_1\bar{A}_2A_2$ . Both these groups  $F = 3^2.2$  are in Table 7.

If  $F/J \cong 2^2$ , we can take  $F = \langle V, i, t \rangle$  and so  $C_G(F)^0 = A_1^2A_1$ , as in Table 7.

Next suppose  $F/J \cong Z_4$ . There is one class of elements of order 4 in  $GL_2(3)$ , with representative  $u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ; this swaps two pairs of  $A_2$  factors, and squares to  $i$ . Hence  $C_G(F)^0 = C_G(V, u)^0 = A_1^2$ , as in Table 7.

If  $F/J \cong Dih_8$ , we can take  $F = \langle V, u, t \rangle$ , and again  $C_G(F)^0 = A_1^2$ .

Now suppose  $F/J \cong Q_8$ . Then  $F/J$  acts transitively on the four  $A_2$  factors, and contains  $i$ , so  $C_G(F)^0 = A_1$ , a diagonal subgroup of  $A_1^4 < A_2^4$ . We claim that  $C_G(F)^0$  is reducible. To see this, observe that  $A_1^4$  centralizes the involution  $i$ ; this involution corresponds to  $w_0$ , the longest element of the Weyl group of  $G$ , and so  $C_G(i) = D_8$ . Now it is easy to check that  $C_G(F)^0 = A_1$  is reducible in this  $D_8$ . Indeed,  $A_1^4$  acts as  $(1, 1, 1, 1)$  on the natural module for  $D_8$  and hence a diagonal subgroup  $A_1$  acts as  $4 + 2^3 + 0^2$ . Thus  $F/J \cong Q_8$  is impossible, and we have now covered all possibilities for  $F/J$ . ■

**Lemma 3.9** *If  $|J|_5 \geq 5$ , then  $F$  is as in Table 7.*

*Proof.* Suppose  $|J|_5 \geq 5$ , and let  $x \in J$  have order 5. As  $C_G(x) = A_4^2$  by Proposition 2.2, there is no element of order 5 in  $C_F(x) \setminus \langle x \rangle$ , and so  $\langle x \rangle$  is a Sylow 5-subgroup of  $F$ .

As  $F$  has no element of order 10 or 15, we have  $C_F(x) = \langle x \rangle$ , and  $|F| = 10$  or 20. By Proposition 2.4,  $N_G(A_4^2)/A_4^2 = \langle t \rangle \cong Z_4$ , where  $t$  interchanges the two  $A_4$  factors and  $t^2$  induces a graph automorphism on both. Hence  $F$  is either  $Dih_{10}$  or  $Frob_{20}$ , and  $C_G(F)^0 = B_2B_2$  or  $B_2$ , respectively, as in Table 7. ■

Lemmas 3.6 – 3.9 cover all cases where the Fitting subgroup  $J$  is nontrivial.

**Lemma 3.10** *Suppose  $J = \text{Fit}(F) = 1$ . Then  $F = \text{Alt}_5$  or  $\text{Sym}_5$  is as in Table 7.*

*Proof.* In this case  $S := \text{soc}(F)$  is a direct product of nonabelian simple groups. As  $5^2$  does not divide  $|F|$ , in fact  $S$  is simple. Proposition 1.2 of [10] shows that  $S \cong \text{Alt}_5$  or  $\text{Alt}_6$ .

Suppose  $S \cong \text{Alt}_5$ . Then  $S$  has subgroups  $D \cong Dih_{10}$  and  $A \cong \text{Alt}_4$ , and by what we have already proved, these subgroups are in Table 7. Hence the involutions in  $S$  are in the class  $2B$  (since those in  $A$  are in this class). If the elements of order 3 in  $S$  are in class  $3A$  (with centralizer  $A_8$ ), then from [5, 3.1] we see that the traces of the elements in  $S$  of orders 2, 3, 5 on  $L(G)$  are  $-8, -4, -2$  respectively, and hence

$$\dim C_{L(G)}(S) = \frac{1}{60}(248 - 8 \cdot 15 - 4 \cdot 20 - 2 \cdot 24) = 0,$$

which is a contradiction. It follows that the elements of order 3 in  $S$  are in the class  $3B$ , with centralizer  $A_2E_6$  and trace 5, so that

$$\dim C_{L(G)}(S) = \frac{1}{60}(248 - 8 \cdot 15 + 5 \cdot 20 - 2 \cdot 24) = 3.$$

Since  $C_G(D)^0 = B_2B_2 < A_4A_4$ , it follows that  $C_G(S)^0 = A_1$ , embedded diagonally and irreducibly in  $A_4A_4$ . Also  $C_G(A_1) = \text{Sym}_5$  by [10, 1.5]. Hence  $F = \text{Alt}_5$  or  $\text{Sym}_5$  and  $C_G(F)^0 = A_1$ , as in Table 7.

Finally, suppose  $S \cong \text{Alt}_6$  and choose a subgroup  $T < S$  with  $T \cong \text{Alt}_5$ . By the above,  $C_G(T)^0 = A_1$  and so  $C_G(S)^0$  must also be  $A_1$ . But as observed before,  $C_G(A_1) = \text{Sym}_5$ , a contradiction. ■

We have now established that  $F$  and  $C_G(F)^0$  must be as in Table 7. To complete the proof of Theorem 1, we need to establish that all these examples exist. This is proved in the following lemma.

**Lemma 3.11** *Let  $F$  and  $C_G(F)^0$  be as in Table 7. Then  $C_G(F)^0$  is  $G$ -irreducible.*

*Proof.* Any subgroup containing a  $G$ -irreducible subgroup is itself  $G$ -irreducible. Thus we need only consider the subgroups  $C_G(F)^0$  for which  $F$  is maximal. These subgroups are given in Table 1; let  $X$  be such a subgroup  $C_G(F)^0$ .

Firstly, if  $X$  has maximal rank then  $X$  is clearly  $G$ -irreducible. For the subgroups not of maximal rank we use the fact that a subgroup with no trivial composition factors on  $L(G)$  is necessarily  $G$ -irreducible (since the Lie algebra of the centre of a Levi subgroup gives a trivial composition factor). It thus remains to show  $X$  has

no trivial composition factors on  $L(E_8)$ . We find the composition factors of  $X$  on  $L(G)$  by restriction from a maximal rank overgroup  $Y$ , as given in the last column of Table 7. The restrictions  $L(G) \downarrow Y$  are given in [12, Lemma 11.2, 11.3] for all of the maximal rank overgroups  $Y$  except for  $A_1A_7$ ,  $A_1^4D_4$  and  $A_2^4$ . The latter subgroups are contained in  $A_1E_7$ ,  $D_4^2$  and  $A_2E_6$ , respectively, and it is straightforward to compute their composition factors on  $L(G)$ .

We finish the proof with two examples of how to calculate the composition factors of  $L(G) \downarrow X$  from those of a maximal rank overgroup  $Y$ . The others all follow similarly and in each case there are no trivial composition factors.

For the first example, let  $X = B_2^3$  so  $p \neq 2$  and  $X$  is contained in the maximal rank overgroup  $D_8$ . From [12, Lemma 11.2],

$$L(G) \downarrow D_8 = V(\lambda_2) + V(\lambda_7),$$

the sum of the exterior square of the natural module for  $D_8$  and a spin module. To find the restriction of the spin module  $V_{D_8}(\lambda_7)$  to  $X$  we consider the chain of subgroups  $X < B_2D_5 < B_2B_5 < D_8$ . By [12, Lemma 11.15(ii)],  $V_{D_8}(\lambda_7) \downarrow B_2B_5 = 01 \otimes \lambda_5$ . Also,  $V_{B_5}(\lambda_5) \downarrow D_5 = \lambda_4 + \lambda_5$  and  $V_{D_5}(\lambda_i) \downarrow B_2^2 = 01 \otimes 01$  for  $i = 4, 5$ . Therefore,

$$L(G) \downarrow X = \bigwedge^2 (10 \otimes 00 \otimes 00 + 00 \otimes 10 \otimes 00 + 00 \otimes 00 \otimes 10 + 0) + (01 \otimes 01 \otimes 01)^2$$

and this has no trivial composition factors.

For the second example, let  $X = A_1D_4$ . Here  $p = 3$  and  $X$  is contained in a maximal rank subgroup  $A_1A_7$ . Then using the restriction  $L(G) \downarrow A_1E_7$  given in [12, Lemma 11.2] we find

$$L(G) \downarrow A_1A_7 = 2 \otimes 0 + 1 \otimes \lambda_2 + 1 \otimes \lambda_6 + 0 \otimes (\lambda_1 + \lambda_7) + 0 \otimes \lambda_4.$$

It is sufficient to show there are no trivial composition factors for  $D_4$  acting on  $V_{A_7}(\lambda)$  for  $\lambda = \lambda_1 + \lambda_7$  and  $\lambda_4$ . By weight considerations, the first module restricts to  $D_4$  as  $V(2\lambda_1) + V(\lambda_2)$  and the second as  $V(2\lambda_3) + V(2\lambda_4)$ . Hence  $L(G) \downarrow X$  has no trivial composition factors.  $\blacksquare$

This completes the proof of Theorem 1 for  $G = E_8$ .

### 3.2 The case $G = E_7$

In this section we prove Theorem 1 for  $G = E_7$ , of adjoint type. Let  $F$  be a finite subgroup of  $G$  such that  $C_G(F)^0$  is  $G$ -irreducible. As before,  $C_G(F)^0$  is semisimple and  $C_G(E)^0$  is  $G$ -irreducible for all nontrivial subgroups  $E$  of  $F$ . Also  $F$  is a  $\{2, 3\}$ -group by Proposition 2.2.

**Lemma 3.12** *If  $F$  is an elementary abelian 2-group, then  $F$  is as in Table 8.*

*Proof.* We may suppose that  $|F| > 2$ . If  $F$  has an element  $e$  in the class  $2B$ , then any further element  $f \in F \setminus \langle e \rangle$  must lie in  $C_G(e) \setminus C_G(e)^0 = A_7.2 \setminus A_7$ , and hence  $F = \langle e, f \rangle \cong 2^2$ ; moreover  $C_G(F)^0 = D_4$ , as in the proof of Lemma 3.2. Hence  $F$  is as in Table 8.

So now suppose that  $F$  is  $2A$ -pure. Let  $1 \neq e \in F$  and  $e_1 \in F \setminus \langle e \rangle$ . Then  $C_G(e) = A_1D_6$ , and diagonalizing in  $SO_{12}$  as in Lemma 3.1, we can take  $e_1 =$

$(-1^4, 1^8)$ . Hence  $C_G(e, e_1)^0 = A_1^3 D_4$ . If there is an element  $e_2 \in E \setminus \langle e, e_1 \rangle$ , then we can take  $e_2 = (1^4, -1^4, 1^4)$ , and so  $C_G(e, e_1, e_2)^0 = A_1^7$ . Both these possibilities are in Table 7, and there are no further possible elements in  $F$ . ■

**Lemma 3.13** *If  $F$  is a 2-group containing an element of order 4, then  $F$  is as in Table 8.*

*Proof.* Let  $e \in F$  of order 4. By Proposition 2.2 we have  $C_G(e)^0 = A_1 A_3^2$ . Suppose  $F \neq \langle e \rangle$ , so there exists  $f \in F$  such that  $e^f = e^{-1}$ . Now  $C_G(e^2) = A_1 D_6$ , and diagonalizing in  $SO_{12}$  as in Lemma 3.1, we may take  $e = (-1^6, 1^6)$  and  $f \in \{f_1, f_2\}$ , where

$$f_1 = (-1, 1^5, -1^3, 1^3), \quad f_2 = (-1^3, 1^3, -1^3, 1^3).$$

If  $f = f_1$  then  $C_G(e, f)^0 = \bar{A}_1 B_1^2 B_2$  and  $\langle e, f \rangle \cong Dih_8$ ; and if  $f = f_2$  then  $C_G(e, f)^0 = \bar{A}_1 B_1^4$  and  $\langle e, f \rangle \cong Q_8$ . Both possibilities are in Table 8. Finally, there are no possible further elements of  $F$ , as can be seen by diagonalizing in the usual way. ■

In view of the previous two lemmas we assume from this point that  $F$  contains an element  $x$  of order 3. Let  $J$  be the Fitting subgroup of  $F$ . Note that  $F$  does not contain an element of order 6 by Proposition 2.2. Therefore  $J$  is a 2-group or a 3-group.

**Lemma 3.14** *If  $J$  is a 3-group then  $F$  and  $C_G(F)^0$  are as given in Table 8.*

*Proof.* Suppose  $|J| = 3$ . If  $|F| = 3$  then by Proposition 2.2 we have  $C_G(F) = A_2 A_5$ . Otherwise  $F \cong Dih_6$  and  $C_G(F)^0 = A_1 C_3$  or  $A_1 A_3$ .

Finally,  $|J| > 3$  is impossible because the centralizer of an element of order 3 in  $A_2 A_5$  is not  $A_2 A_5$ -irreducible. ■

We may now assume that  $J$  is a 2-group. By Lemmas 3.12 and 3.13,  $J$  is as in Table 8 and the action of  $x$  shows that the only possibilities are  $J \cong 2^2, 2^3$  or  $Q_8$ .

**Lemma 3.15** *If  $J \cong 2^2$  then  $F$  and  $C_G(F)^0$  are as given in Table 8.*

*Proof.* Suppose  $C_G(J)^0 = A_1^3 D_4$ . By Proposition 2.4,  $N_G(A_1^3 D_4)/A_1^3 D_4 \cong Sym_3$  acting simultaneously on both the  $A_1^3$  and the  $D_4$  factors. Therefore  $C_G(J, x)^0 = A_1 A_2$  or  $A_1 G_2$  with  $\langle J, x \rangle \cong Alt_4$ . The subgroup  $A_1 G_2$  is  $A_1 D_6$ -reducible by Proposition 2.7, and therefore does not appear in Table 8. If  $F \neq \langle J, x \rangle$  then we must have  $F \cong Sym_4$  with  $C_G(F)^0 = A_1 A_1$ .

Now suppose  $C_G(J)^0 = D_4 < A_7$ . By [7, Lemma 2.15], we have  $N_G(D_4)/(D_4 \times C_G(D_4)) \cong Sym_3$ . Therefore  $C_G(J, x)^0 = A_2$  or  $G_2$ . The subgroup  $G_2$  is  $A_7$ -reducible and therefore does not appear in Table 8. If  $F \neq \langle J, x \rangle$  then  $F \cong Sym_4$  with  $C_G(F)^0 = A_1$ . ■

**Lemma 3.16** *There are no possible subgroups  $F$  with  $J \cong 2^3$  or  $Q_8$ .*

*Proof.* Suppose  $J \cong 2^3$  so  $C_G(J)^0 = A_1^7$ . By Proposition 2.4 we have  $N_G(A_1^7)/A_1^7 \cong GL_3(2)$ . The element  $x \in F$  therefore acts as a product of two



disjoint 3-cycles on the seven  $A_1$  factors. But the centralizer  $C_G(J, x)^0$  is then  $A_1 D_6$ -reducible by an argument in the first paragraph of the proof of Lemma 3.6.

Finally, if  $J \cong Q_8$  and  $C_G(J)^0 = A_1 B_1^4$ , then  $C_G(J, x)^0 = A_1 B_1 B_1$  which is clearly  $A_1 D_6$ -reducible. ■

The proof of Theorem 1 for  $G = E_7$  is now complete, apart from showing that all the subgroups  $C_G(F)^0$  in Table 8 are  $G$ -irreducible. This is proved in similar fashion to Lemma 3.11.

### 3.3 The case $G = \text{Aut } E_6$

Let  $G = \text{Aut } E_6 = E_6.2$ , and let  $F$  be a finite subgroup of  $G$  such that  $C_{G'}(F)^0$  is  $G'$ -irreducible.

**Lemma 3.17** *If  $F$  has an element  $x$  of order 4, then  $F = \langle x \rangle$  and  $C_{G'}(F)^0 = A_1 A_3$ .*

*Proof.* By Proposition 2.3,  $C_G(x)^0 = A_1 A_3$ . By Proposition 2.7,  $A_1 A_3$  contains no proper  $A_1 A_5$ -irreducible connected subgroups and therefore  $F = \langle x \rangle$  as claimed. ■

We now assume that  $F$  has no element of order 4.

**Lemma 3.18** *If  $F$  is an elementary abelian 2-group then it appears in Table 10.*

*Proof.* If  $|F| = 2$  then  $F$  and  $C_G(F)^0$  are as in Table 10 by Proposition 2.2. Now suppose  $F = \langle t, u \rangle \cong 2^2$ . Then  $C_G(t)^0 = A_1 A_5, F_4$  or  $C_4$  and therefore  $C_G(F)^0 = A_1 A_3, A_1 C_3, B_4$  or  $C_2^2$ . The  $A_1 A_3$  case is ruled out by Lemma 3.17. The  $B_4$  and  $C_2^2$  subgroups are both contained in  $D_5$ -parabolic subgroups. Therefore  $C_G(F)^0 = A_1 C_3$ . Finally, if  $F$  has a further involution  $v$  then  $C_G(F)^0 = A_1^2 C_2$ , which by Proposition 2.7 is  $A_1 A_5$ -reducible. ■

We now let  $J$  be the Fitting subgroup of  $F$ . Since  $F$  is a  $\{2, 3\}$ -group,  $J$  is nontrivial.

**Lemma 3.19** *If  $J$  is not a 2-group or a 3-group, then  $F$  is as in Table 10.*

*Proof.* Under the assumptions of the lemma,  $J$  has an element  $x$  of order 6. Then  $C_G(x)^0 = A_2 A_2$  by Proposition 2.3. If  $F \neq \langle x \rangle$  then there exists an element  $t \in F \setminus J$  inverting  $x$  with  $t^2 \in J$ . Since  $F$  has no element of order 4 we have  $t^2 = 1$  and  $\langle J, t \rangle \cong Dih_{12}$ . The element  $t$  induces a graph automorphism on both  $A_2$  factors and so  $C_G(J, t)^0 = A_1 A_1$ . Since the two factors are non-conjugate, there is no element swapping them and hence  $F = \langle J, t \rangle$ . ■

**Lemma 3.20** *If  $J$  is a 2-group then  $F = J$ .*

*Proof.* The possibilities for the 2-group  $J$  are in Table 10, from which we see that no element of order 3 can act as a graph automorphism on  $C_G(J)^0$ . ■

**Lemma 3.21** *If  $J$  is a 3-group then  $F$  is as in Table 10.*

*Proof.* Suppose  $J = \langle x \rangle \cong Z_3$ . If  $F = J$  then  $C_G(F)^0 = A_2^3$  by Proposition 2.2. Otherwise,  $F = \langle J, t \rangle \cong Dih_6$  where  $t$  is an involution in  $N_G(A_2^3)/A_2^3 \cong 2 \times S_3$  by Proposition 2.4. This gives two possibilities for  $t$ . If  $t$  is the central involution then  $t$  induces a graph automorphism on each factor  $A_2$  and  $C_G(J, t)^0 = A_1^3$ . If  $t$  is not central then  $C_G(J, t)^0 = A_1 A_1$ .

Now suppose  $|J| > 3$  and let  $x, y \in J$  with  $\langle x, y \rangle \cong 3^2$ . Then  $C_G(x) = A_2^3.3$  and  $y$  cyclically permutes the three  $A_2$  factors, so  $C_G(x, y)^0$  is a diagonal subgroup  $A_2$ . However, the elements in class  $3A$  have trace  $-3$  on  $L(G)$  and so

$$\dim C_{L(G)}(S) = \frac{1}{9}(78 - 8 \cdot 3) = 6,$$

a contradiction. ■

Finally, we need to prove that all the subgroups  $C_G(F)^0$  in Table 10 are  $G'$ -irreducible.

**Lemma 3.22** *Let  $F$  and  $C_G(F)^0$  be as in Table 9. Then  $C_G(F)^0$  is  $G'$ -irreducible.*

*Proof.* This is proved in a similar fashion to Lemma 3.11 for most of the subgroups. Specifically, all of the subgroups have no trivial composition factors on  $L(G)$  except for  $A_1 A_5$ ,  $\bar{A}_1 C_3$  and  $\bar{A}_1 A_3$  when  $p = 3$ , all of which have exactly one trivial composition factor. There are no Levi subgroups of  $G'$  containing a subgroup of type  $A_1 A_5$  or  $A_1 C_3$ . Hence both are  $G'$ -irreducible.

Now consider  $X = A_1 A_3$ . Assume  $X$  is  $G'$ -reducible and choose a minimal parabolic subgroup  $P$  containing  $X$ . By [11, Theorem 1],  $X$  is contained in a Levi subgroup  $L$  of  $P$  and by minimality  $X$  is  $L$ -irreducible. Hence  $L = D_5 T_1$  or  $A_1 A_3 T_2$ , where  $T_i$  denotes a central torus of rank  $i$ . The second possibility is ruled out since  $X$  has only one trivial composition factor on  $L(G)$ . So  $X$  is an irreducible subgroup of  $L' = D_5$ . The  $A_1$  factor of  $X$  is generated by root groups of  $D_5$  and so  $C_{D_5}(A_1)^0 = A_1 A_3$ . Thus  $C_{D_5}(X)^0$  contains a subgroup  $A_1$ , contradicting the  $L$ -irreducibility of  $X$ . Hence  $X$  is  $G'$ -irreducible, as required. ■

This completes the proof of Theorem 1 for  $G = \text{Aut } E_6$ .

### 3.4 The case $G = F_4$

Let  $G = F_4$ , and let  $F$  be a finite subgroup of  $G$  such that  $C_G(F)^0$  is  $G$ -irreducible.

**Lemma 3.23** *If  $F$  is an elementary abelian 2-group, then  $F$  and  $C_G(F)^0$  are given in Table 11.*

*Proof.* If  $F \cong Z_2$  then  $C_G(F) = B_4$  or  $A_1 C_3$  by Proposition 2.2. Now suppose  $F = \langle t, u \rangle \cong 2^2$ . Then  $u \in C_G(t) = B_4$  or  $A_1 C_3$ . Therefore  $C_G(F)^0 = D_4$  or  $A_1^2 B_2$ . Now suppose  $|F| > 4$ . A  $2^2$  subgroup of  $F$  must contain a  $2A$  involution, say  $t$ , with centralizer  $B_4$ . Then  $B_4/\langle t \rangle \cong SO_9$  and the image of  $F$  in  $SO_9$  is elementary abelian by Proposition 2.5. Since  $C_G(F)^0$  is  $G$ -irreducible, it follows that the image is  $\langle u, v \rangle \cong 2^2$  with  $u = (-1^8, 1)$  and  $v = (-1^4, 1^5)$ . Therefore  $C_{SO_9}(F)^0 = SO_4 SO_4$ , and so  $C_G(F)^0 = A_1^4$ . ■

**Lemma 3.24** *If  $F$  is a 2-group containing an element  $x$  of order 4, then  $F$  and  $C_G(F)^0$  are given in Table 11.*

*Proof.* By Proposition 2.2,  $C_G(x) = A_1A_3$ . Now suppose  $|F| > 4$ . Since  $A_1A_3$  is not contained in  $A_1C_3$  it follows that  $C_G(x^2) = B_4$ . Therefore  $F/\langle x^2 \rangle$  is elementary abelian in  $SO_9$ . Diagonalizing as before we may assume  $x = (1^3, -1^6)$ . Since  $C_G(F)^0$  is  $G$ -irreducible, it must be the case that  $|F/\langle x^2 \rangle| = 4$  and a further involution in  $F$  is either  $u_1 = (-1^6, 1^3)$  or  $u_2 = (1^5, -1^4)$ . In the first case the order of  $xu_1$  is 4 and hence  $F \cong Q_8$  with  $C_G(F)^0 = B_1^3$ . In the second case the order of  $xu_2$  is 2 and hence  $F \cong Dih_8$  with  $C_G(F)^0 = B_1B_2$ . ■

Now let  $J$  be the Fitting subgroup of  $F$ . Since  $F$  has no element of order 6 it follows that  $J$  is either a 2-group or a 3-group.

**Lemma 3.25** *If  $J$  is a 2-group then  $F$  and  $C_G(F)^0$  are given in Table 11.*

*Proof.* By the previous two lemmas we may assume that  $F$  is not a 2-group, hence contains an element  $x$  of order 3. The only possibilities for  $J$  are  $2^2$ ,  $2^3$  or  $Q_8$ , with  $C_G(J)^0 = D_4$ ,  $A_1^4$  or  $B_1^3$ , respectively. The last two cases are ruled out since any proper diagonal connected subgroup of  $A_1^4$  or  $B_1^3$  such that each projection involves no nontrivial field automorphisms is not  $B_4$ -irreducible by Proposition 2.7.

Hence  $J \cong 2^2$  and  $C_G(J)^0 = D_4$ . If  $F = \langle J, x \rangle \cong Alt_4$ , then  $C_G(F)^0 = A_2$  or  $G_2$ ; and  $G_2$  is not possible since it is contained in a Levi subgroup of type  $B_3$ . And if  $F \neq \langle J, x \rangle$  then  $F \cong Sym_4$  and  $C_G(F)^0 = A_1$ . ■

**Lemma 3.26** *If  $J$  is a 3-group then  $F$  and  $C_G(F)^0$  are given in Table 11.*

*Proof.* Let  $J = \langle x \rangle$ , so  $C_G(x) = A_2A_2$ . Proposition 2.4 gives  $N_G(A_2A_2)/A_2A_2 = \langle t \rangle \cong Z_2$ , where  $t$  acts as a graph automorphism on each factor. Therefore  $F = \langle x, t \rangle \cong Dih_6$  with  $C_G(F)^0 = A_1A_1$ . ■

As before, the fact that all the subgroups  $C_G(F)^0$  in Table 11 are  $G$ -irreducible is proved in similar fashion to Lemma 3.11; in particular they all have no trivial composition factors on  $L(G)$ .

This completes the proof of Theorem 1 for  $G = F_4$ .

### 3.5 The case $G = G_2$

**Lemma 3.27** *Let  $F$  be a finite subgroup of  $G = G_2$  such that  $C_G(F)^0$  is  $G$ -irreducible. Then  $F$  and  $C_G(F)^0$  are as in Table 12.*

*Proof.* By Proposition 2.2, nonidentity elements of  $F$  have order 2 or 3. If  $F$  is a 2-group then  $F$  contains an involution  $t$  with  $C_G(t) = A_1A_1$ ; the centralizer of an involution in  $A_1A_1$  is reducible and therefore  $F = \langle t \rangle$ . Similarly, if  $F$  is a 3-group then  $F = \langle u \rangle \cong 3$  and  $C_G(F) = A_2$ . The only remaining possibility is  $F = \langle t, u \rangle \cong Dih_6$ . Since  $N_G(A_2)/A_2 \cong 2$ , such an example exists and  $C_G(F)^0 = A_1$ . ■

This completes the proof of Theorem 1.

## 4 Proof of Proposition 3

In this section we prove the following generalisation of Proposition 3.

**Proposition 4.1** *Let  $G$  be a classical simple adjoint algebraic group in characteristic  $p \geq 0$  with natural module  $V$ , and let  $H = \text{Aut } G$ . Suppose  $F$  is a finite subgroup of  $H$  such that  $C_G(F)^0$  is  $G$ -irreducible. Then  $F$  is an elementary abelian 2-group (or a group of order 3 or 6 in the case where  $G = D_4$ ), and one of the following holds.*

(i)  $G = PSL_n$ ,  $F \cap G = 1$ ,  $|F| = 2$  and

*if  $n$  is even, then  $C_G(F)^0 = PSp_n$  or  $PSO_n$  ( $p \neq 2$ );*

*if  $n$  is odd, then  $p \neq 2$  and  $C_G(F)^0 = PSO_n$ .*

(ii)  $G = H = PSp_{2n}$ ,  $p \neq 2$ , and taking preimages in  $Sp_{2n}$ ,

$$C_{Sp_{2n}}(F)^0 = \prod_i Sp_{2n_i} = \prod_i Sp(W_i),$$

*where  $\sum n_i = n$  and  $W_i$  are the distinct weight spaces of  $F$  on  $V$ .*

(iii)  $G = PSO_n$  ( $n \neq 8$ ),  $H = PO_n$ ,  $p \neq 2$ , and taking preimages in  $O_n$ ,

$$C_{O_n}(F)^0 = \prod_i SO_{n_i} = \prod_i SO(W_i),$$

*where  $n_i \geq 3$  for all  $i$ ,  $\sum n_i = n$  or  $n - 1$ , and  $W_i$  are weight spaces of  $F$ .*

(iv)  $G = PSO_{2n}$  ( $n \neq 4$ ),  $p = 2$ ,  $F \cap G = 1$ ,  $|F| = 2$  and  $C_G(F)^0 = SO_{2n-1}$ .

(v)  $G = D_4 = PSO_8$ ,  $H = D_4.Sym_3$ , and  $F$ ,  $C_G(F)^0$  are as in Table 6.

*Proof.* First suppose  $G = PSL_n$ . If  $F \cap G \neq 1$  then  $C_G(F \cap G)^0$  is reducible, so  $F \cap G = 1$ . Hence  $|F| = 2$  and now the conclusion in part (i) follows from Proposition 2.1. Similarly, if  $G = D_n = PSO_{2n}$  with  $n \neq 4$  and  $p = 2$  (so that  $H = G.2$ ), then  $F \cap G = 1$ ,  $|F| = 2$  and  $C_G(F)^0 = B_{n-1}$  by Proposition 2.1, as in (iv).

Now suppose  $G = PSp_{2n}$ . Then  $H = G$  and the centralizer in  $G$  of any element of order greater than 2 is reducible. Hence  $F$  is an elementary abelian 2-group and  $p \neq 2$ . The preimage  $\hat{F}$  of  $F$  in  $Sp_n$  must also be elementary abelian, and if we let  $W_i$  ( $1 \leq i \leq k$ ) be the weight spaces of  $\hat{F}$  on  $V$ , then  $V = W_1 \perp \cdots \perp W_k$  and  $C_{Sp_{2n}}(\hat{F}) = \prod Sp(W_i)$ , as in conclusion (ii).

A similar proof applies when  $G = PSO_n$  with  $n \neq 8$  and  $p \neq 2$ , giving (iii).

It remains to handle  $G = D_4 = PSO_8$ . Here  $H = G.Sym_3$ . If  $F \leq PO_8 = G.2$  then the above proof shows that  $F = 2$  or  $2^2$  is as in Table 6. Now suppose 3 divides  $|F|$ , so that  $F$  contains an element  $x$  of order 3 inducing a triality automorphism on  $G$ . By Proposition 2.1,  $C_G(x) = G_2$  or  $A_2$ , with  $p \neq 3$  in the latter case.

If there is an element  $y \in C_F(x) \setminus \langle x \rangle$ , then  $y \in G_2$  or  $A_2$  has irreducible centralizer, which forces  $y$  to be an involution in  $G_2$ . So in this case  $F = \langle x, y \rangle \cong Z_6$  and  $C_G(F) = C_{G_2}(y) = \bar{A}_1 A_1$ , as in Table 6. This subgroup has composition factors of dimensions 1, 3 and 4 on  $V$ , so is  $G$ -irreducible.

Table 6:  $G = \text{Aut } D_4$ : finite subgroups  $F$  with irreducible centralizer

$F$	$F \cap G^0$	$C_G(F)^0$
2	2	$A_1^4 (p \neq 2)$
	1	$B_3$
	1	$B_1 B_2 (p \neq 2)$
$2^2$	2	$A_1^2 B_1 (p \neq 2)$
3	1	$G_2$
	1	$A_2 (p \neq 3)$
6	2	$A_1 A_1 (p \neq 2)$
$Dih_6$	1	$G_2$
		$A_1 (p \neq 2, 3)$

We may now suppose that  $C_F(x) = \langle x \rangle$  and  $F \neq \langle x \rangle$ . This implies that  $F = \langle x, t \rangle \cong Dih_6$ . If  $C_G(x) = G_2$  then  $t$  must centralize  $G_2$ , so that  $C_G(F) = G_2$ . And if  $C_G(x) = A_2$  then  $t$  induces a graph automorphism on  $A_2$  (see Proposition 2.1), so  $C_G(F) = A_1$  and  $p \neq 2$ , as in Table 6. This completes the proof. ■

## 5 Tables of results

This section consists of the tables referred to in Theorem 1.

## References

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Table 7:  $G = E_8$ : finite subgroups  $F$  with irreducible centralizer

$F$	$C_G(F)^0$	elements of $F$	maximal rank overgp. of $C_G(F)^0$
2	$A_1 E_7$	$2A$	
	$D_8$	$2B$	
$2^2$	$D_4^2$	$2B^3$	
	$A_1^2 D_6$	$2A^2, 2B$	
$2^3$	$A_1^4 D_4$	$2A^4, 2B^3$	
	$A_1^8$	$2B^7$	
$2^4$	$A_1^8$	$2A^8, 2B^7$	
4	$A_1 A_7$	$2A, 4A^2$	
	$A_3 D_5$	$2B, 4B^2$	
$4 \times 2$	$A_1^2 A_3^2$	$2A^2, 2B, 4B^4$	
$Dih_8$	$B_1^2 B_4$	$2A^4, 2B, 4B^2$	$D_8$
	$B_1^2 B_2^2$	$2B^5, 4B^2$	$D_8$
	$B_1 B_2 B_3$	$2A^2, 2B^3, 4B^2$	$D_8$
$Q_8$	$A_1 D_4$	$2A, 4A^6$	$A_1 A_7$
	$B_2^3$	$2B, 4B^6$	$D_8$
	$B_1^3 B_3$	$2B, 4B^6$	$D_8$
$Dih_8 \times 2$	$\bar{A}_1^2 B_1^2 B_2$	$2A^6, 2B^5, 4B^4$	$D_8$
$4 \circ Dih_8$	$A_3 B_1^3$	$2A^6, 2B, 4B^8$	$D_8$
$Q_8 \times 2$	$\bar{A}_1^2 B_1^4$	$2A^2, 2B, 4B^{12}$	$D_8$
$2_-^{1+4}$	$B_1^5$	$2A^{10}, 2B, 4B^{20}$	$D_8$
3	$A_2 E_6$	$3B^2$	
	$A_8$	$3A^2$	
$3^2$	$A_2^4$	$3B^8$	
5	$A_4^2$	$5A$	
6	$A_1 A_2 A_5$	$2A, 3B^2, 6A^2$	
$Dih_6$	$A_1 F_4$	$2A^3, 3B^2$	$A_2 E_6$
	$A_1 C_4$	$2B^3, 3B^2$	$A_2 E_6$
	$B_4$	$2B^3, 3A^2$	$A_8$
$Dih_{12}$	$\bar{A}_1 A_1 C_3$	$2A^4, 2B^3, 3B^2, 6A^2$	$A_1 A_2 A_5$
$G_{12}$	$\bar{A}_1 A_1 A_3$	$2A, 3B^2, 4A^6, 6A^2$	$A_1 A_2 A_5$
$Alt_4$	$A_2 G_2$	$2B^3, 3B^8$	$D_4^2$
	$A_2 A_2$	$2B^3, 3A^8$	$D_4^2$
$Sym_4$	$A_1 G_2$	$2A^6, 2B^3, 3B^8, 4B^6$	$D_4^2$
	$A_1 A_1$	$2B^9, 3A^8, 4B^6$	$D_4^2$
$Alt_4 \times 2$	$\bar{A}_1 A_1 A_2$	$2A^4, 2B^3, 3B^8, 6A^8$	$A_1^4 D_4$
$Sym_4 \times 2$	$\bar{A}_1 A_1 A_1$	$2A^{12}, 2B^7, 3B^8, 4B^{12}, 6A^8$	$A_1^4 D_4$
$SL_2(3)$	$\bar{A}_1 A_2$	$2A, 3B^8, 4A^6, 6A^8$	$A_1 A_7$
$3^2.2$	$A_1^4$	$2B^9, 3B^8$	$A_2^4$
	$A_1 A_2 A_2$	$2A^3, 3B^8, 6A^6$	$A_2^4$
$3^2.4$	$A_1^2$	$2B^9, 3B^8, 4B^{18}$	$A_2^4$
$3^2.2^2$	$A_1^2 A_1$	$2A^6, 2B^9, 3B^8, 6A^{12}$	$A_2^4$
$3^2.Dih_8$	$A_1^2$	$2A^{12}, 2B^9, 3B^8, 4B^{18}, 6A^{24}$	$A_2^4$
$Dih_{10}$	$B_2^2$	$2B^5, 5A^4$	$A_4^2$
$Frob_{20}$	$B_2$	$2B^5, 4B^{10}, 5A^4$	$A_4^2$
$Alt_5$	$A_1$	$2B^{15}, 3B^{20}, 5A^{24}$	$A_4^2$
$Sym_5$	$A_1$	$2A^{10}, 2B^{15}, 3B^{20}, 4B^{30}, 5A^{24}, 6A^{20}$	$A_4^2$

Table 8:  $G = E_7$  (adjoint): finite subgroups  $F$  with irreducible centralizer

$F$	$C_G(F)^0$	elements of $F$	maximal rank overgp. of $C_G(F)^0$
2	$A_1 D_6$	$2A$	$A_7$
	$A_7$	$2B$	
$2^2$	$A_1^3 D_4$	$2A^3$	
	$D_4$	$2B^3$	
$2^3$	$A_1^7$	$2A^7$	
4	$A_1 A_3^2$	$2A, 4A^2$	$A_1 D_6$
$Dih_8$	$\bar{A}_1 B_1^2 B_2$	$2A^5, 4A^2$	
$Q_8$	$\bar{A}_1 B_1^4$	$2A, 4A^6$	
3	$A_2 A_5$	$3A$	$A_2 A_5$
$Dih_6$	$A_1 C_3$	$2A^3, 3A^2$	
	$A_1 A_3$	$2B^3, 3A^2$	
$Alt_4$	$A_1 A_2$	$2A^3, 3A^8$	
	$A_2$	$2B^3, 3A^8$	
$Sym_4$	$A_1 A_1$	$2A^9, 3A^8, 4A^6$	$A_1^3 D_4$

 Table 9:  $G = E_6$ : finite subgroups  $F$  with irreducible centralizer

$F$	$C_G(F)^0$	elements of $F$	maximal rank overgp. of $C_G(F)^0$
2	$A_1 A_5$	$2A$	$A_2^3$
3	$A_2^3$	$3A^2$	
$Dih_6$	$A_1 A_1$	$2A^3, 3A^2$	

 Table 10:  $G = \text{Aut } E_6$ : finite subgroups  $F$  with irreducible centralizer

$F$	$F \cap G^0$	$C_G(F)^0$	elements of $F$	maximal rank overgp. of $C_G(F)^0$
2	2	$A_1 A_5$	$2A$	$A_1 A_5$
	1	$F_4$	$2B$	
	1	$C_4$	$2C$	
4	2	$\bar{A}_1 A_3$	$2A, 4A^2$	
$2^2$	2	$\bar{A}_1 C_3$	$2A, 2B, 2C$	
3	3	$A_2^3$	$3A^2$	$A_2^3$
$Dih_6$	$Dih_6$	$A_1 A_1$	$2A^3, 3A^2$	
$Dih_{12}$	$Dih_6$	$A_1 A_1$	$2A^3, 2B, 2C^3, 3A^2, 6A^2$	

Table 11:  $G = F_4$ : finite subgroups  $F$  with irreducible centralizer

$F$	$C_G(F)^0$	elements of $F$	maximal rank overgp. of $C_G(F)^0$
2	$B_4$	$2A$	
	$A_1C_3$	$2B$	
$2^2$	$A_1^2C_2$	$2A, 2B^2$	
	$D_4$	$2A^3$	
$2^3$	$A_1^4$	$2A^3, 2B^4$	
4	$A_1A_3$	$2A, 4A^2$	
$Dih_8$	$B_1B_2$	$2A^3, 2B^2, 4A^2$	
$Q_8$	$B_1^3$	$2A, 4A^6$	
3	$A_2A_2$	$3A^2$	
$Dih_6$	$A_1A_1$	$2B^3, 3A^2$	
$Alt_4$	$A_2$	$2A^3, 3A^8$	$D_4$
$Sym_4$	$A_1$	$2A^3, 2B^6, 3A^8, 4A^6$	$D_4$

Table 12:  $G = G_2$ : finite subgroups  $F$  with irreducible centralizer

$F$	$C_G(F)^0$	elements of $F$	maximal rank overgp. of $C_G(F)^0$
2	$A_1A_1$	$2A$	$A_2$
3	$A_2$	$3A^2$	
$Dih_6$	$A_1$	$2A^3, 3A^2$	



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